

Conformal field theory and integrable systems associated to elliptic curves

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1 Introduction

It has become clear over the years that quantum groups (i.e., quasitriangular Hopf algebras, see [D]) and their semiclassical counterpart, Poisson Lie groups, are an essential algebraic structure underlying three related subjects: integrable models of statistical mechanics, conformal field theory and integrable models of quantum field theory in 1+1 dimensions. Still, some points remain obscure from the point of view of Hopf algebras. In particular, integrable models associated with elliptic curves are still poorly understood. We propose here an elliptic version of quantum groups, based on the relation to conformal field theory, which hopefully will be helpful to complete the picture.

But before going to the elliptic case, let us remind the relations between the three subjects in the simpler rational and trigonometric cases.

In integrable models of statistical mechanics (see [B],[F]), the basic object is an R -matrix, i.e., a meromorphic function of a spectral parameter $z \in \mathbf{C}$ with values in $\text{End}(V \otimes V)$ for some vector space V , obeying of the Yang–Baxter equation

$$R^{(12)}(z)R^{(13)}(z+w)R^{(23)}(w) = R^{(23)}(w)R^{(13)}(z+w)R^{(12)}(z),$$

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in $\text{End}(V \otimes V \otimes V)$. The notation is customary in this subject: $X^{(j)} \in \text{End}(V \otimes \cdots \otimes V)$, for $X \in \text{End}(V)$, means $\text{Id} \otimes \cdots \otimes \text{Id} \otimes X \otimes \text{Id} \otimes \cdots \otimes \text{Id}$, with X at the j th place, and if $R = \Sigma X_\nu \otimes Y_\nu$, $R^{(ij)} = \Sigma X_\nu^{(i)} Y_\nu^{(j)}$.

The Yang–Baxter equation implies the commutativity of infinitely many transfer matrices constructed out of R . Rational and trigonometric solutions of the Yang–Baxter equation appear naturally in the theory of Quasitriangular Hopf algebras.

If R depends on a parameter \hbar so that $R = \text{Id} + \hbar r + O(\hbar^2)$, as $\hbar \rightarrow 0$, then the “classical r -matrix” r obeys the classical Yang–Baxter equation

$$[r^{(12)}(z), r^{(13)}(z+w) + r^{(23)}(w)] + [r^{(13)}(z+w), r^{(23)}(w)] = 0. \quad (1)$$

This equation appears in the theory of Poisson–Lie groups, but has the following relation with conformal field theory. In the skew-symmetric case $r(z) = -r^{(21)}(-z)$, it is the compatibility condition for the system of equations

$$\partial_{z_i} u = \sum_{j: j \neq i} r^{(ij)}(z_i - z_j) u \quad (2)$$

for a function $u(z_1, \dots, z_n)$ on $\mathbf{C}^n - \cup_{i < j} \{z | z_i = z_j\}$, with values in $V \otimes \cdots \otimes V$. In the rational case, very simple skew-symmetric solutions are known: $r(z) = C/z$, where $C \in \mathfrak{g} \otimes \mathfrak{g}$ is a symmetric invariant tensor of a finite dimensional Lie algebra \mathfrak{g} acting on a representation space V . The corresponding system of differential equations is the Knizhnik–Zamolodchikov (KZ) equation for conformal blocks of the Wess–Zumino–Witten model of conformal field theory on the sphere. Solutions of (1) in $\mathfrak{g} \otimes \mathfrak{g}$, for simple Lie algebras \mathfrak{g} , were partially classified by Belavin and Drinfeld [BD], who in particular proved that, under a non-degeneracy assumption, solutions can be divided into three classes, according, say, to the lattice of poles: rational, trigonometric and elliptic. Elliptic solutions are completely classified and exist only for sl_N .

Recently, Frenkel and Reshetikhin [FR] considered the “quantization” of the Knizhnik–Zamolodchikov equations based on the representation theory of Yangians (rational case) and affine quantum enveloping algebras (trigonometric case). They are a compatible system of *difference* equations which as $\hbar \rightarrow 0$ reduce to the differential equations (2). In an important special case, these difference equations had been introduced earlier by Smirnov [S] who derived them as equations for “form factors” in integrable quantum field theory, and gave relevant solutions. In the quantum field theory setting R has the interpretation of two-particle scattering matrix, and is required to obey the “unitarity” relation $R(z)R^{(21)}(-z) = \text{Id}$, as well as a “crossing symmetry” condition.

In the elliptic case, one knows solutions of the Yang–Baxter equation whose semiclassical limit are the sl_N solutions discussed above [Bel]. The relevant algebraic structure is here the Sklyanin algebra [S], [Ch], which however does not fall into the general quantum group theory. Although elliptic solutions related to other Lie algebras have not been found (and they could not have

a semiclassical limit by the Belavin–Drinfeld theorem), many elliptic solutions of the Star–Triangle relation, a close cousin of the Yang–Baxter equation, are known (see [JMO], [JKMO], [DJKMO]). This is somewhat mysterious, as, in the trigonometric case, solutions of both equations are in one-to-one correspondence. Another apparent puzzle we want to point out, is that conformal field theory can be defined on arbitrary Riemann surfaces [TUY] whereas r -matrices exist only up to genus one, and in genus one only for sl_N .

We will start from the solution to this last puzzle to arrive, via quantization and difference equations to a notion of elliptic quantum group, which is the algebraic structure underlying the elliptic solutions of the Star–Triangle relation.

Let us mention some other recent progress in similar directions. In [FR] solutions of the Star–Triangle relation are obtained as connection matrices for the trigonometric quantum KZ equations. Very recently, Foda et al. [FIJKMY] have proposed an elliptic quantum algebra of non-zero level, by another modification of the Yang–Baxter equation.

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2 Conformal field theory, KZB equations

Our starting point is the set of genus one Knizhnik–Zamolodchikov–Bernard (KZB) equations, obtained by Bernard [B1, B2] as generalization of the KZ equations. These equations have been studied recently in [FG], [EK], [FW].

Let \mathfrak{g} be a simple complex Lie algebra with invariant bilinear form normalized in such a way that long roots have square length 2. Fix a Cartan subalgebra \mathfrak{h} . The KZB equations are equations for a function $u(z_1, \dots, z_n, \tau, \lambda)$ with values in the weight zero subspace (the subspace killed by \mathfrak{h}) of a tensor product of irreducible finite dimensional representations of \mathfrak{g} . The arguments z_1, \dots, z_n, τ are complex numbers with τ in the upper half plane, and the z_i are distinct modulo the lattice $\mathbf{Z} + \tau\mathbf{Z}$, and $\lambda \in \mathfrak{h}$. Let us introduce coordinates $\lambda = \sum \lambda_\nu h_\nu$ in terms of an orthonormal basis (h_ν) of \mathfrak{h} . In the formulation of [FW], the KZB equations take the form

$$\kappa \partial_{z_j} u = - \sum_{\nu} h_{\nu}^{(j)} \partial_{\lambda_{\nu}} u + \sum_{l: l \neq j} \Omega^{(j,l)}(z_j - z_l, \tau, \lambda) u, \quad (3)$$

$$4\pi i \kappa \partial_{\tau} u = \sum_{\nu} \partial_{\lambda_{\nu}}^2 u + \sum_{j,l} H^{(j,l)}(z_j - z_l, \tau, \lambda) u, \quad (4)$$

Here κ is an integer parameter which is large enough depending on the representations in the tensor product and $\Omega, H \in \mathfrak{g} \otimes \mathfrak{g}$ are tensors preserving the weight zero subspace that we now describe. Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$ be the root decomposition of \mathfrak{g} , and $C \in S^2 \mathfrak{g}$ be the symmetric invariant tensor dual to the

invariant bilinear form on \mathfrak{g} . Write $C = \sum_{\alpha \in \Delta \cup \{0\}} C_\alpha$, where $C_0 = \sum_\nu h_\nu \otimes h_\nu$ and $C_\alpha \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$. Let $\theta_1(t, \tau)$ be Jacobi's theta function

$$\theta_1(t|\tau) = - \sum_{j=-\infty}^{\infty} e^{\pi i(j+\frac{1}{2})^2 \tau + 2\pi i(j+\frac{1}{2})(t+\frac{1}{2})}.$$

and introduce functions ρ, σ :

$$\begin{aligned} \rho(t) &= \partial_t \log \theta_1(t|\tau), \\ \sigma(w, t) &= \frac{\theta_1(w-t|\tau) \partial_t \theta_1(0|\tau)}{\theta_1(w|\tau) \theta_1(t|\tau)}. \end{aligned}$$

The tensor Ω is given by

$$\Omega(z, \tau, \lambda) = \rho(z) C_0 + \sum_{\alpha \in \Delta} \sigma(\alpha(\lambda), z) C_\alpha$$

The tensor H has a similar form. We need the following special functions of $t \in \mathbf{C}$, expressed in terms of $\sigma(w, t)$, $\rho(t)$ and Weierstrass' elliptic function \wp with periods $1, \tau$.

$$\begin{aligned} I(t) &= \frac{1}{2}(\rho(t)^2 - \wp(t)), \\ J_w(t) &= \partial_t \sigma(w, t) + (\rho(t) + \rho(w)) \sigma(w, t). \end{aligned}$$

These functions are regular at $t = 0$. The tensor H is then given by the formula

$$H(t, \tau, \lambda) = I(t) C_0 + \sum_{\alpha \in \Delta} J_{\alpha(\lambda)}(t) C_\alpha.$$

As shown in [FW], the functions u from conformal field theory have a special dependence on the parameter λ . For fixed z, τ , the function u , as a function of λ , belongs to a finite dimensional space of antiinvariant theta function of level κ . Therefore the right way of looking at these equation is to consider u as a function of z_1, \dots, z_n, τ taking values in a finite dimensional space of functions of λ .

The tensors have the skew-symmetry property $\Omega(z) + \Omega^{(21)}(-z) = 0$, and $H(z) - H^{(21)}(-z) = 0$ and commute with $X^{(1)} + X^{(2)}$ for all $X \in \mathfrak{h}$. The compatibility condition of (3) is then the *modified classical Yang-Baxter equation* [FW]

$$\begin{aligned} \sum_\nu \partial_{\lambda_\nu} \Omega^{(1,2)} h_\nu^{(3)} + \sum_\nu \partial_{\lambda_\nu} \Omega^{(2,3)} h_\nu^{(1)} + \sum_\nu \partial_{\lambda_\nu} \Omega^{(3,1)} h_\nu^{(2)} \\ - [\Omega^{(1,2)}, \Omega^{(1,3)}] - [\Omega^{(1,2)}, \Omega^{(2,3)}] - [\Omega^{(1,3)}, \Omega^{(2,3)}] = 0 \end{aligned} \quad (5)$$

in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. In this equation, $\Omega^{(ij)}$ is taken at $(z_i - z_j, \tau, \lambda)$. Moreover, there are relations involving H , which we do not consider here, as we will consider only the first equation (3). The quantization of (4) is an important open problem related, for $n = 1$, to the theory of elliptic Macdonald polynomials [EK].

3 The quantization

Let \mathbf{h} be the complexification of a Euclidean space \mathbf{h}_r and extend the scalar product to a bilinear form on \mathbf{h} . View \mathbf{h} as an Abelian Lie algebra. We consider finite dimensional diagonalizable \mathbf{h} -modules V . This means that we have a weight decomposition $V = \bigoplus_{\mu \in \mathbf{h}} V[\mu]$ such that $\lambda \in \mathbf{h}$ acts as (μ, λ) on $V[\mu]$. Let $P_\mu \in \text{End}(V)$ be the projection onto $V[\mu]$.

It is convenient to introduce the following notation. Suppose V_1, \dots, V_n are finite dimensional diagonalizable \mathbf{h} -modules. If $f(\lambda)$ is a meromorphic function on \mathbf{h} with values in $\bigotimes_i V_i = V_1 \otimes \dots \otimes V_n$ or $\text{End}(\bigotimes_i V_i)$, and η_i are complex numbers, we define a function on \mathbf{h}

$$f(\lambda + \sum \eta_i h^{(i)}) = \sum_{\mu_1, \dots, \mu_n} \prod_{i=1}^n P_{\mu_i}^{(i)} f(\lambda + \sum \eta_i \mu_i),$$

taking values in the same space as f .

Given \mathbf{h} and V as above, the quantization of (5) is an equation for a meromorphic function R of the spectral parameter $z \in \mathbf{C}$ and an additional variable $\lambda \in \mathbf{h}$, taking values in $\text{End}(V \otimes V)$

$$\begin{aligned} & R^{(12)}(z_{12}, \lambda + \eta h^{(3)}) R^{(13)}(z_{13}, \lambda - \eta h^{(2)}) R^{(23)}(z_{23}, \lambda + \eta h^{(1)}) = \\ & = R^{(23)}(z_{23}, \lambda - \eta h^{(1)}) R^{(13)}(z_{13}, \lambda + \eta h^{(2)}) R^{(12)}(z_{12}, \lambda - \eta h^{(3)}). \end{aligned} \quad (6)$$

The parameter η is proportional to \hbar , and z_{ij} stands for $z_i - z_j$. This equation forms the basis for the subsequent analysis. Let us call it modified Yang–Baxter equation (MYBE). Note that a similar equation, without spectral parameter, has appeared for the monodromy matrices in Liouville theory, see [GN], [B], [AF]. We supplement it by the “unitarity” condition

$$R^{(12)}(z_{12}, \lambda) R^{(21)}(z_{21}, \lambda) = \text{Id}_{V \otimes V}, \quad (7)$$

and the “weight zero” condition

$$[X^{(1)} + X^{(2)}, R(z, \lambda)] = 0, \quad \forall X \in \mathbf{h}. \quad (8)$$

We say that $R \in \text{End}(V \otimes V)$ is a *generalized quantum R-matrix* if it obeys (6), (7), (8).

If we have a family of solutions parametrized by η in some neighborhood of the origin, and $R(z, \lambda) = \text{Id}_{V \otimes V} - 2\eta\Omega(z, \lambda) + O(\eta^2)$ has a “semiclassical asymptotic expansion”, then (6) reduces to the modified classical Yang–Baxter equation (5).

Here are examples of solutions. Take \mathbf{h} to be the Abelian Lie algebra of diagonal N by N complex matrices, with bilinear form $\text{Trace}(AB)$, acting on $V = \mathbf{C}^N$. Denote by E_{ij} the N by N matrix with a one in the i th row and j th column and zeroes everywhere else. Then we have

Proposition 3.1 *The function*

$$R(z, \lambda) = \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} \frac{\sigma(\gamma, \lambda_{ij})}{\sigma(\gamma, z)} E_{ii} \otimes E_{jj} + \sum_{i \neq j} \frac{\sigma(\lambda_{ij}, z)}{\sigma(\gamma, z)} E_{ij} \otimes E_{ji},$$

is a “unitary” weight zero solution of the modified Yang–Baxter equation, i.e., it is a generalized quantum R -matrix, with $\eta = \gamma/2$.

The proof is based on comparing poles and behavior under translation of spectral parameters by $\mathbf{Z} + \tau\mathbf{Z}$ on both sides of the equation. It uses unitarity, the \mathbf{Z} periodicity of R and the transformation property

$$\begin{aligned} R(z + \tau, \lambda) &= e^{-4\pi i \eta} \exp(2\pi i(\eta C_0 + \lambda^{(1)})) R(z, \lambda) \exp(2\pi i(\eta C_0 - \lambda^{(1)})), \\ C_0 &= \sum E_{ii} \otimes E_{ii}. \end{aligned} \quad (9)$$

Two limiting cases of this solution are of interest. First, if $\tau \rightarrow i\infty$ and $\lambda_j - \lambda_l \rightarrow i\infty$, if $j < l$, we recover the well known trigonometric R matrix connected with the quantum enveloping algebra of $A_{N-1}^{(1)}$ (see [J]).

The semiclassical limit is more subtle. To obtain precisely Ω of the KZB equations, replace $\sigma(\gamma, \lambda_{ij})$ by $\sigma(\gamma, \lambda_{ij}) \exp(\rho(\lambda_{ij})\gamma)$. It turns out that this replacement is compatible with the MYBE (but violates the assumption of meromorphy). Then $-\gamma R$ has a semiclassical asymptotic expansion with Ω (for gl_N) as coefficient of $-\gamma$.

Following the Leningrad school (see [F]), one associates a bialgebra with quadratic relations to each solution of the Yang–Baxter equation. In our case we have modify slightly the construction. Let us consider an “algebra” $A(R)$ associated to a generalized quantum R -matrix R , generated by meromorphic functions on \mathbf{h} and the matrix elements (in some basis of V) of a matrix $L(u, \lambda) \in \text{End}(V)$ with non commutative entries, subject to the relations

$$\begin{aligned} R^{(12)}(z_{12}, \lambda + \eta h) L^{(1)}(z_1, \lambda - \eta h^{(2)}) L^{(2)}(z_2, \lambda + \eta h^{(1)}) &= \\ = L^{(2)}(z_2, \lambda - \eta h^{(1)}) L^{(1)}(z_1, \lambda + \eta h^{(2)}) R^{(12)}(z_{12}, \lambda - \eta h). \end{aligned}$$

Instead of giving a more precise definition of this algebra, let us define the more important notion (for our purposes) of representation of $A(R)$.

Definition: Let $R \in \text{End}(V \otimes V)$ be a meromorphic unitary weight zero solution of the MYBE (a generalized quantum R -matrix). A representation of $A(R)$ is a diagonalizable \mathbf{h} -module W together with a meromorphic function $L(u, \lambda)$ (called L -operator) on $\mathbf{C} \times \mathbf{h}$ with values in $\text{End}(V \otimes W)$ such that the identity

$$\begin{aligned} R^{(12)}(z_{12}, \lambda + \eta h^{(3)}) L^{(13)}(z_1, \lambda - \eta h^{(2)}) L^{(23)}(z_2, \lambda + \eta h^{(1)}) &= \\ = L^{(23)}(z_2, \lambda - \eta h^{(1)}) L^{(13)}(z_1, \lambda + \eta h^{(2)}) R^{(12)}(z_{12}, \lambda - \eta h^{(3)}) \end{aligned}$$

holds in $\text{End}(V \otimes V \otimes W)$, and so that L is of weight zero:

$$[X^{(1)} + X^{(2)}, L(u, \lambda)] = 0, \quad \forall X \in \mathbf{h}.$$

We have natural notions of homomorphisms of representations.

Theorem 3.2 (*Existence and coassociativity of the coproduct*) *Let (W, L) and (W', L') be representations of $A(R)$. Then $W \otimes W'$ with \mathbf{h} -module structure $X(w \otimes w') = Xw \otimes w' + w \otimes Xw'$ and L -operator*

$$L^{(12)}(z, \lambda + \eta h^{(3)}) L^{(13)}(z, \lambda - \eta h^{(2)})$$

is a representation of $A(R)$. Moreover, if we have three representation W, W', W'' , then the representations $(W \otimes W') \otimes W''$ and $W \otimes (W' \otimes W'')$ are isomorphic (with the obvious isomorphism).

Note also that if $L(z, \lambda)$ is an L -operator then also $L(z - w, \lambda)$ for any complex number w . Since the MYBE and the weight zero condition mean that (V, R) is a representation, we may construct representations on $V^{\otimes n} = V \otimes \cdots \otimes V$ by iterating the construction of Theorem 3.2. The corresponding L operator is the “monodromy matrix” with parameters z_1, \dots, z_n :

$$\prod_{j=2}^{n+1} R^{(1j)}(z - z_j, \lambda - \eta \sum_{1 < i < j} h^{(i)} + \eta \sum_{j < i \leq n+1} h^{(i)}).$$

(the factors are ordered from left to right). Although the construction is very reminiscent of the Quantum Inverse Scattering Method [F], we cannot at this point construct commuting transfer matrices by taking the trace of the monodromy matrices. As will be explained below, one has to pass to IRF models.

4 Difference equations

We now give the quantum version of the KZB equation (3). As in the trigonometric case [S, FR] it is a system of difference equations. The system is symmetric, i.e., there is an action of the symmetric group mapping solutions to solutions. In the trigonometric and rational case, symmetric meromorphic solutions with proper pole structure are “form factors” of integrable models of Quantum Field Theory in two dimensions [S].

It is convenient to formulate the construction in terms of representation theory of the affine symmetric group. Let S_n be the symmetric group acting on \mathbf{C}^n by permutations of coordinates, and $s_j, j = 1, \dots, n-1$ be the transpositions $(j, j+1)$. These transpositions generate S_n with relations $s_j s_l = s_l s_j$, if $|j-l| \geq 2$, $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$, and $s_j^2 = 1$. Let also $P \in \text{End}(V \otimes V)$ be the “flip” operator $Pu \otimes v = v \otimes u$ and if R is a generalized quantum R -matrix, set $\hat{R} = RP$. The defining properties of a generalized quantum R -matrix imply:

Proposition 4.1 *Suppose that R is a generalized quantum R -matrix. The formula*

$$s_j f(z, \lambda) = \hat{R}^{(j, j+1)}(z_{j, j+1}, \lambda - \eta \sum_{i < j} h^{(i)} + \eta \sum_{i > j+1} h^{(i)}) f(s_j z, \lambda) \quad (11)$$

defines a representation of S_n on meromorphic functions on $\mathbf{C} \times \mathbf{h}$ with values in $V^{\otimes n}$.

The (extended) affine symmetric group S_n^a is the semidirect product of S_n by Z_n . It is generated by s_j and commuting generators e_j , $j = 1, \dots, n$, with relations $s_j e_l = e_l s_j$, if $l \neq j, j+1$ and $s_j e_j = e_{j+1} s_j$. Let us introduce a parameter $a \in \mathbf{C}$ and let e_j act on $z \in \mathbf{C}^n$ as $e_j(z_1, \dots, z_n) = (z_1, \dots, z_j - a, \dots, z_n)$. Note that S_n^a is actually generated by s_1, \dots, s_{n-1} and e_n , since the other e_j are constructed recursively as $e_j = s_j e_{j+1} s_j$.

Theorem 4.2 *Suppose that R is a generalized quantum R -matrix. Let $T_j f(z, \lambda) = f(z, \lambda - 2\eta h^{(j)})$, $\Gamma_j f(z, \lambda) = f(z + a, \lambda)$ and $R^{(j,n)}$ denote the operator of multiplication by $R^{(j,n)}(z_{j,n}, \lambda - \eta \sum_{i < j} h^{(i)} + \eta \sum_{j < i < n} h^{(i)})$. Then (11) and*

$$e_n f = R^{(n-1,n)} \dots R^{(2,n)} R^{(1,n)} \Gamma_n T_n f$$

define a representation of S_n^a on meromorphic functions on $\mathbf{C} \times \mathbf{h}$ with values in $V^{\otimes n}$.

It is easy to calculate the action of the other generators e_j . One gets expressions similar to the ones in [FR, S].

The compatible system of difference equations (Quantum KZB equations) is then

$$e_j f = f, \quad j = 1, \dots, n.$$

The symmetric group maps solutions to solutions.

Moreover, it turns out that, for special values of a , the representation of S_n^a for the solution of Prop. 3.1 preserves a space of theta functions, as in the classical case.

5 IRF models

In our setting, the relation between the generalized quantum R -matrix and the Boltzmann weights W of the corresponding interaction-round-a-face (IRF) model [B] is very simple. Let $R \in \text{End}(V \otimes V)$ be a generalized quantum R -matrix, and let $V[\mu]$ be the component of weight $\mu \in \mathbf{h}^*$ of V , with projection $E[\mu] : V \rightarrow V[\mu]$. Then for $a, b, c, d \in \mathbf{h}^*$, such that $b - a$, $c - b$, $d - a$ and $c - d$ occur in the weight decomposition of V , define a linear map

$$W(a, b, c, d, z, \lambda) : V[d - a] \otimes V[c - d] \rightarrow V[c - b] \otimes V[b - a],$$

by the formula

$$W(a, b, c, d, z, \lambda) = E[c - b] \otimes E[b - a] R(z, \lambda - \eta a - \eta c) |_{V[d-a] \otimes V[c-d]}. \quad (12)$$

Note that $W(a + x, b + x, c + x, d + x, z, \lambda + 2\eta x)$ is independent of $x \in \mathbf{h} \simeq \mathbf{h}^*$. Set $W(a, b, c, d, z) = W(a, b, c, d, z, 0)$.

Theorem 5.1 *If R is a solution of the MYBE, then $W(a, b, c, d, z)$ obeys the Star-Triangle relation*

$$\begin{aligned} & \sum_g W(b, c, d, g, z_{12})^{(12)} W(a, b, g, f, z_{13})^{(13)} W(f, g, d, e, z_{23})^{(23)} \\ &= \sum_g W(a, b, c, g, z_{23})^{(23)} W(g, c, d, e, z_{13})^{(13)} W(a, g, e, f, z_{12})^{(12)}, \end{aligned}$$

on $V[f - a] \otimes V[e - f] \otimes V[d - e]$.

The familiar form of the Star-Triangle relation [B],[JMO] is recovered when the spaces $V[\mu]$ are 1-dimensional. Upon choice of a basis, the Boltzmann weights $W(a, b, c, d, z)$ are then numbers.

For example, if R is the solution of Prop. 3.1, we obtain the well-known $A_{n-1}^{(1)}$ solution (see [JMO], [JKMO] and references therein).

It is known that solutions of the Star-Triangle relations can be used to construct solvable models of statistical mechanics. Several elliptic solutions are known. It is to be expected that the representation theory of the algebra $A(R)$ above will give a more systematic theory of solutions. Also, the fact that these Boltzmann weights arise as connection matrices of the quantum KZ equation [FR] and the similarity of our Yang–Baxter equation with the triangle equation of [GN], suggest that our algebra is the quantum conformal field theory analogue of $U_q(\mathfrak{g})$, the algebra governing the monodromy of conformal field theory.

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